Pearson New International Edition

Fundamentals of Engineering Electromagnetics

David K. Cheng

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Edinburgh Gate
Harlow
Essex CM20 2JE
England and Associated Companies throughout the world

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ISBN 10: 1-292-02658-8
ISBN 13: 978-1-292-02658-9

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library

Printed in the United States of America

Preface

This book is designed for use as an undergraduate text on engineering electromagnetics. Electromagnetics is one of the most fundamental subjects in an electrical engineering curriculum. Knowledge of the laws governing electric and magnetic fields is essential to the understanding of the principle of operation of electric and magnetic instruments and machines, and mastery of the basic theory of electromagnetic waves is indispensible to explaining action-at-a-distance electromagnetic phenomena and systems.

Because most electromagnetic variables are functions of three-dimensional space coordinates as well as of time, the subject matter is inherently more involved than electric circuit theory, and an adequate coverage normally requires a sequence of two semester-courses, or three courses in a quarter system. However, some electrical engineering curricula do not schedule that much time for electromagnetics. The purpose of this book is to meet the demand for a textbook that not only presents the fundamentals of electromagnetism in a concise and logical manner, but also includes important engineering application topics such as electric motors, transmission lines, waveguides, antennas, antenna arrays, and radar systems.

I feel that one of the basic difficulties that students have in learning electromagnetics is their failure to grasp the concept of an electromagnetic model. The traditional inductive approach of starting with experimental laws and gradually synthesizing them into Maxwell's equations tends to be fragmented and incohesive; and the introduction of gradient, divergence, and curl operations appears to be ad hoc and arbitrary. On the other hand, the extreme of starting with the entire set of Maxwell's equations, which are of considerable complexity, as fundamental postulates is likely to cause consternation and resistance in students at the outset. The question of the necessity and sufficiency of these general equations is not addressed, and the concept of the electromagnetic model is left vague.

This book builds the electromagnetic model using an *axiomatic*

approach in steps[†]—first for static electric fields, then for static magnetic fields, and finally for time-varying fields leading to Maxwell's equations. The mathematical basis for each step is Helmholtz's theorem, which states that a vector field is determined to within an additive constant if both its divergence and its curl are specified everywhere. A physical justification of this theorem may be based on the fact that the divergence of a vector field is a measure of the strength of its flow source and the curl of the field is a measure of the strength of its vortex source. When the strengths of both the flow source and the vortex source are specified, the vector field is determined.

For the development of the electrostatic model in free space, it is only necessary to define a single vector (namely, the electric field intensity **E**) by specifying its divergence and its curl as postulates. All other relations in electrostatics for free space, including Coulomb's law and Gauss's law, can be derived from the two rather simple postulates. Relations in material media can be developed through the concept of equivalent charge distributions of polarized dielectrics.

Similarly, for the magnetostatic model in free space it is necessary to define only a single magnetic flux density vector **B** by specifying its divergence and its curl as postulates; all other formulas can be derived from these two postulates. Relations in material media can be developed through the concept of equivalent current densities. Of course, the validity of the postulates lies in their ability to yield results that conform with experimental evidence.

For time-varying fields, the electric and magnetic field intensities are coupled. The curl **E** postulate for the electrostatic model must be modified to conform with Faraday's law. In addition, the curl **B** postulate for the magnetostatic model must also be modified in order to be consistent with the equation of continuity. We have, then, the four Maxwell's equations that constitute the electromagnetic model. I believe that this gradual development of the electromagnetic model based on Helmholtz's theorem is novel, systematic, pedagogically sound, and more easily accepted by students.

A short Chapter 1 of the book provides some motivation for the study of electromagnetism. It also introduces the source functions, the fundamental field quantities, and the three universal constants for free space in the electromagnetic model. Chapter 2 reviews the basics of vector algebra, vector calculus, and the relations of Cartesian, cylindrical, and spherical coordinate systems. Chapter 3 develops the governing laws and methods of solution of electrostatic problems. Chapter 4 is on steady electric current fields and resistance calculations. Chapter 5 deals with static magnetic fields. Chapter 6, on time-varying electromagnetic fields, starts with Faraday's law of

[†]D. K. Cheng, "An alternative approach for developing introductory electromagnetics," *IEEE Antennas and Propagation Society Newsletter*, pp. 11–13, Feb. 1983.

electromagnetic induction, and leads to Maxwell's equations and wave equations. The characteristics of plane electromagnetic waves are treated in Chapter 7. The theory and applications of transmission lines are studied in Chapter 8. Further engineering applications of electromagnetic fields and waves are discussed in Chapter 9 (waveguides and cavity resonators) and Chapter 10 (antennas, antenna arrays, and radar systems). Much of the material has been adapted and reduced from my larger book, *Field and Wave Electromagnetics*[†], but in this book I have incorporated a number of innovative pedagogical features.

Each chapter of this book starts with an overview section that provides qualitative guidance to the topics to be discussed in the chapter. Throughout the book worked-out examples follow important formulas and quantitative relations to illustrate methods for solving typical problems. Where appropriate, simple exercises with answers are included to test students' ability to handle related situations. At irregular intervals, a group of review questions are inserted after several related sections. These questions serve to provide an immediate feedback of the topics just discussed and to reinforce students' qualitative understanding of the material. Also, a number of pertinent remarks usually follow the review questions. These remarks contain some points of special importance that the students may have overlooked. When new definitions, concepts, or relations are introduced, short notes are added in the margins to emphasize their significance. At the end of each chapter there is a summary with bulleted items summarizing the main topics covered in the chapter. I hope that these pedagogical aids will prove to be useful in helping students learn electromagnetics and its applications.

Many dedicated people, besides the author, are involved in the publication of a book such as this one. I wish to acknowledge the interest and support of Senior Editor Eileen Bernadette Moran and Executive Editor Don Fowley since the inception of this project. I also wish to express my appreciation to Production Supervisor Helen Wythe for her friendly assistance in keeping the production on schedule, as well as to Roberta Lewis, Amy Willcutt, Laura Michaels, and Alena Konecny for their contributions. Jim and Rosa Sullivan of Tech-Graphics were responsible for the illustrations. To them I offer my appreciation for their fine work. Above all, I wish to thank my wife, Enid, for her patience, understanding and encouragement through all phases of my challenging task of completing this book.

D.K.C.

[†]D. K. Cheng, *Field and Wave Electromagnetics*, 2nd ed., Addison-Wesley, Reading, Mass., 1989.

An Introductory Note to the Student

This book is your guide as you embark on the journey of learning engineering electromagnetics. Two questions may immediately come to mind: What is electromagnetics and why is it important? A short answer to the first question is that electromagnetics is the study of the effects of electric charges at rest or in motion. It is important because electromagnetic theory is essential in explaining electromagnetic phenomena and in understanding the principle of operation and the characteristics of electric, magnetic, and electromagnetic engineering devices. Contemporary society relies heavily on electromagnetic devices and systems. Think, for example, of microwave ovens, cathode-ray oscilloscopes, radio, television, radar, satellite communication, automatic instrument-landing systems, and electromagnetic energy conversion (motors and generators).

The basic principles of electromagnetism have been known for over one hundred fifty years. To study a mature scientific subject in an organized and logical way it is necessary to establish a valid theoretical model, which usually consists of a few basic quantities and some fundamental postulates (hypotheses, or axioms). Other relations and consequences are then developed from these postulates. For instance, the study of classical mechanics is based on a theoretical model that defines the quantities mass, velocity, acceleration, force, momentum, and energy. The fundamental postulates of the model are Newton's laws of motion, conservation of momentum, and conservation of energy. These postulates cannot be derived from other theorems; but all other relations and formulas in non-relativistic mechanics (situations where the velocity of motion is negligible compared to the velocity of light) can be developed from these postulates.

Similarly, in our study of electromagnetics we need first to establish an electromagnetic model. Chapter 1 of this book defines the basic quantities of our electromagnetic model. The fundamental postulates are introduced in gradual steps as they are needed when we deal with static electric fields, static

magnetic fields, and time-varying fields in separate chapters. Various theorems and other results are then derived from these postulates. Engineering applications of the principles and methods developed throughout the text are explored further in the last several chapters.

In order to express our postulates and derive useful results in succinct forms, we must have the proper mathematical tools. In electromagnetics we most often encounter vectors, quantities that have both a magnitude and a direction. Hence, we must be proficient in vector algebra and vector calculus. These are covered in Chapter 2 on Vector Analysis. We must not only acquire a facility in manipulating vectors, but also understand the physical meaning of the various operations involving vectors. A deficiency in vector analysis in the study of electromagnetism is similar to a deficiency in algebra and calculus in the study of physics. Proficiency in the use of mathematical tools is essential for obtaining fruitful results.

Most likely, you have already studied circuit theory. Circuit theory deals with lumped-parameter systems consisting of components characterized by lumped parameters as resistances, inductances, and capacitances. Voltages and currents are the main system variables. For d-c circuits the system variables are constants, and the governing equations are algebraic equations. The system variables in a-c circuits are time-dependent; they are scalar quantities and are independent of space coordinates. The governing equations are ordinary differential equations. On the other hand, most electromagnetic variables are functions of time as well as of space coordinates. Many are vectors. Even in static cases the governing equations are, in general, partial differential equations. But partial differential equations can be broken up into ordinary differential equations, which you have encountered in courses on physics and linear system analysis. In simple situations where symmetries exist, partial differential equations reduce to ordinary differential equations. The separation of time and space dependence is achieved by the use of phasors.

Because of the need for defining more quantities and using more mathematical manipulations in electromagnetics, you may initially get the impression that electromagnetic theory is abstract. In fact, electromagnetic theory is no more abstract than circuit theory in the sense that the validity of both can be verified by experimentally measured results. We simply have to do more work in order to develop a logical and complete theory that can explain a wider variety of phenomena. The challenge of electromagnetic theory is not in the abstractness of the subject matter but rather in the process of mastering the electromagnetic model and the associated rules of operation.

You will find that each chapter of this book starts with an OVERVIEW section that introduces the topics to be discussed in the chapter. As new definitions, concepts, or relations are introduced, short notes appear in the margins to draw your attention to them. At the end of some related sections,

at irregular intervals, are REVIEW QUESTIONS, which serve to provide an immediate feedback of the topics just discussed and to reinforce your qualitative understanding of the material. You should be able to answer these questions with confidence. If not, you should go back to these sections and clear up your doubts. A REMARKS box usually follows the review questions. It contains some points of special importance that you may have overlooked, but that you should understand and remember. At the conclusion of each chapter there is a SUMMARY section that itemizes the more important results obtained in the chapter. Its function is to emphasize the significance of these results without repeating the mathematical formulas.

Throughout the book new terms and important statements are printed in ***boldface italic***, and the more important formulas are boxed. Worked-out examples are provided to illustrate methods for solving typical problems. Where appropriate, simple exercises with answers are included. You should do these exercises as they occur, so that you can see if you have mastered the basic quantitative skills just presented. The problems at the end of a chapter are used to extend what you have learned from the chapter and to test your ability in tackling new situations. Answers to odd-numbered problems, included at the end of the book, give you a self-check and reassurance on your progress.

The learning of electromagnetics is an intellectual journey; this book is your guide, but you must bring along your dedication and perseverance. As you explore the territory of engineering electromagnetics, we hope you will have a stimulating and rewarding experience.

The author.

Learning is not attained by chance;
it must be sought for with ardor and attended to with diligence.

—Abigail Adams
(in letter to John Quincy Adams, 1780)

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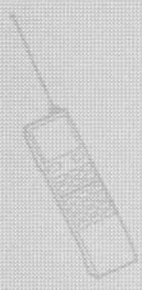


CHAPTER 1

Definition of electromagnetics

1-1 OVERVIEW

Electromagnetics is the study of the electric and magnetic phenomena caused by electric charges at rest or in motion. The existence of electric charges was discovered more than two and a half milleniums ago by a Greek astronomer and philosopher Thales of Miletus. He noted that an amber rod, after being rubbed with silk or wool, attracted straw and small bits of paper. He attributed this mysterious property to the amber rod. The Greek word for amber is *elektron*, from which was derived the words *electron*, *electronics*, *electricity*, and so on.

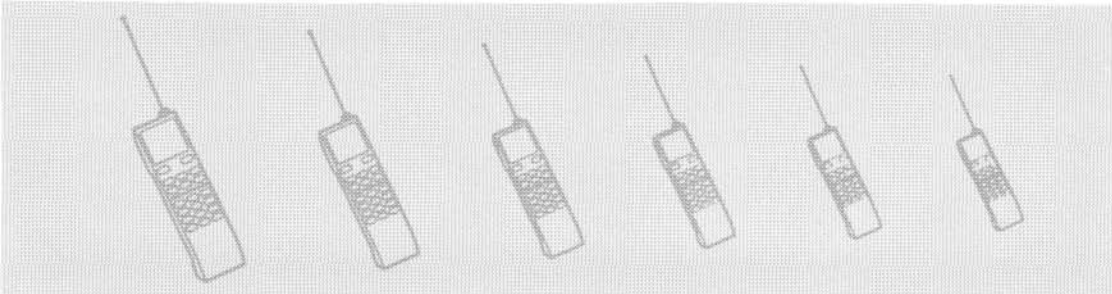
Two kinds of charges: positive and negative

Field: a spatial distribution of a quantity

From elementary physics we know that there are two kinds of charges: positive and negative. Both positive and negative charges are sources of an electric field. Moving charges produce a current, which gives rise to a magnetic field. Here we tentatively speak of electric field and magnetic field in a general way; more definitive meanings will be attached to these terms later. A ***field*** is a spatial distribution of a quantity, which may or may not be a function of time. A time-varying electric field is accompanied by a magnetic field, and vice versa. In other words, time-varying electric and magnetic fields are coupled, resulting in an electromagnetic field. Under certain conditions, time-dependent electromagnetic fields produce waves that radiate from the source.

Fields and waves help explain action at a distance.

The concept of fields and waves is essential in the explanation of action at a distance. For instance, we learned from elementary mechanics that masses attract each other. This is why objects fall toward the Earth's surface. But since there are no elastic strings connecting a free-falling object and the Earth, how do we



The Electromagnetic Model

explain this pheomenon? We explain this action-at-a-distance phenomenon by postulating the existence of a gravitational field. Similarly, the possibilities of satellite communication and of receiving signals from space probes millions of miles away can be explained only by postulating the existence of electric and magnetic fields and electromagnetic waves. In this book, *Fundamentals of Engineering Electromagnetics*, we study the fundamental laws of electromagnetism and some of their engineering applications.

Circuit theory cannot explain mobile phone communication.

A simple situation will illustrate the need for electromagnetic-field concepts. Figure 1-1 depicts a mobile telephone with an attached antenna. On transmit, a source at the base feeds the antenna with a message-carrying current at an appropriate carrier frequency. From a circuit-theory point of view, the source feeds into an open circuit because the upper tip of the antenna is not connected to anything physically; hence no current would flow, and nothing would happen. This viewpoint, of course, cannot explain why communication can be established between moving telephone units. Electromagnetic concepts must be used. We shall see in Chapter 10 that when the length of the antenna is an appreciable part of the carrier wavelength, a nonuniform current will flow along the open-ended antenna. This current radiates a time-varying electromagnetic field in space, which propagates as an electromagnetic wave and induces currents in other antennas at a distance. The message is then detected in the receiving unit.

Constructing a model

In this first chapter we begin the task of constructing an electromagnetic model, from which we shall develop the subject of engineering electromagnetics.

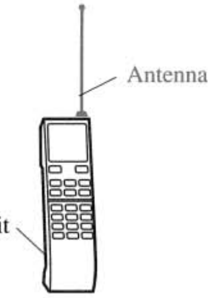


FIGURE 1-1 A mobile telephone.

1-2 THE ELECTROMAGNETIC MODEL

Inductive and deductive approaches

There are two approaches in the development of a scientific subject: the inductive approach and the deductive approach. Using the inductive approach, one follows the historical development of the subject, starting with the observations of some simple experiments and inferring from them laws and theorems. It is a process of reasoning from particular phenomena to general principles. The deductive approach, on the other hand, postulates a few fundamental relations for an idealized model. The postulated relations are axioms, from which particular laws and theorems can be derived. The validity of the model and the axioms is verified by their ability to predict consequences that check with experimental observations. In this book we prefer to use the deductive or axiomatic approach because it is more concise and enables the development of the subject of electromagnetics in an orderly way.

Three essential steps are involved in building a theory based on an idealized model.

Steps for developing a theory from an idealized model

STEP 1 Define some basic quantities germane to the subject of study.

STEP 2 Specify the rules of operation (the mathematics) of these quantities.

STEP 3 Postulate some fundamental relations. (These postulates or laws are usually based on numerous experimental observations acquired under controlled conditions and synthesized by brilliant minds.)

The circuit model

A familiar example is the circuit theory built on a *circuit model* of ideal sources and pure resistances, inductances, and capacitances. In this case the basic quantities are voltages (V), currents (I), resistances (R), inductances (L), and capacitances (C); the rules of operations are those of algebra, ordinary differential equations, and Laplace transformation; and the fundamental postulates are Kirchhoff's voltage and current laws. Many relations and formulas can be derived from this basically rather simple model, and the

responses of very elaborate networks can be determined. The validity and value of the model have been amply demonstrated.

The three steps for developing an electromagnetic theory from an electromagnetic model

In a like manner, an electromagnetic theory can be built on a suitably chosen electromagnetic model. In this section we shall take the first step of defining the basic quantities of electromagnetics. The second step, the rules of operation, encompasses vector algebra, vector calculus, and partial differential equations. The fundamentals of vector algebra and vector calculus will be discussed in Chapter 2 (Vector Analysis), and the techniques for solving partial differential equations will be introduced when these equations arise later in the book. The third step, the fundamental postulates, will be presented in three substeps as we deal with static electric fields, static magnetic fields, and electromagnetic fields, respectively.

Basic quantities in the electromagnetic model: source quantities and field quantities

The quantities in our electromagnetic model can be divided roughly into two categories: source quantities and field quantities. The source of an electromagnetic field is invariably electric charges at rest or in motion. However, an electromagnetic field may cause a redistribution of charges, which will, in turn, change the field; hence the separation between the cause and the effect is not always so distinct.

Electric charges

We use the symbol q (sometimes Q) to denote ***electric charge***. Electric charge is a fundamental property of matter and exists only in positive or negative integral multiples of the charge on an electron, $-e$.

$$\boxed{e = 1.60 \times 10^{-19} \qquad \text{(C)},} \tag{1-1}$$

where C is the abbreviation of the unit of charge, coulomb.† It is named after the French physicist Charles A. de Coulomb, who formulated Coulomb's law in 1785. (Coulomb's law will be discussed in Chapter 3.) A coulomb is a very large unit for electric charge; it takes $1/(1.60 \times 10^{-19})$ or 6.25 million trillion electrons to make up -1(C). In fact, two 1-(C) charges 1(m) apart will exert a force of approximately 1 million tons on each other. Some other physical constants for the electron are listed in Appendix B–2.

Unit of charge: coulomb (C)

The principle of ***conservation of electric charge***, like the principle of conservation of energy, is a fundamental postulate or law of physics. It states that electric charge is conserved; that is, it can neither be created nor be destroyed. This is a law of nature and cannot be derived from other principles or relations.

Electric charges can move from one place to another and can be redistributed under the influence of an electromagnetic field; but the algebraic sum of the positive and negative charges in a closed (isolated) system remains

†The system of units will be discussed in Section 1–3.

Conservation of electric charge is a fundamental postulate of physics.

unchanged. ***The principle of conservation of electric charge must be satisfied at all times and under any circumstances.*** Any formulation or solution of an electromagnetic problem that violates the principle of conservation of electric charge *must be* incorrect.

Although, in a microscopic sense, electric charge either does or does not exist at a point in a discrete manner, these abrupt variations on an atomic scale are unimportant when we consider the electromagnetic effects of large aggregates of charges. In constructing a macroscopic or large-scale theory of electromagnetism we find that the use of smoothed-out average density functions yields very good results. (The same approach is used in mechanics where a smoothed-out mass density function is defined, in spite of the fact that mass is associated only with elementary particles in a discrete manner on an atomic scale.) We define a ***volume charge density***, ρ_v, as a source quantity as follows:

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} \qquad (\text{C/m}^3), \tag{1-2}$$

Volume, surface, and line charge densities— average densities in the macroscopic sense

where Δq is the amount of charge in a very small volume Δv. How small should Δv be? It should be small enough to represent an accurate variation of ρ_v but large enough to contain a very large number of discrete charges. For example, an elemental cube with sides as small as 1 micron (10^{-6} m or 1 μm) has a volume of 10^{-18}(m^3), which will still contain about 10^{11} (100 billion) atoms. A smoothed-out function of space coordinates, ρ_v, defined with such a small Δv is expected to yield accurate macroscopic results for nearly all practical purposes.

In some physical situations an amount of charge Δq may be identified with an element of surface Δs, or an element of line $\Delta \ell$. In such cases it will be more appropriate to define a ***surface charge density***, ρ_s, or a ***line charge density***, ρ_ℓ:

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} \qquad (\text{C/m}^2), \tag{1-3}$$

$$\rho_\ell = \lim_{\Delta \ell \to 0} \frac{\Delta q}{\Delta \ell} \qquad (\text{C/m}). \tag{1-4}$$

Charge densities are point functions.

Except for certain special situations, charge densities vary from point to point; hence ρ_v, ρ_s, and ρ_ℓ are, in general, *point functions* of space coordinates.

Current is the rate of change of charge with respect to time; that is,

$$I = \frac{dq}{dt} \qquad (\text{C/s or A}), \tag{1-5}$$

where I itself may be time-dependent. The unit of current is coulomb per second (C/s), which is the same as ampere (A). A current must flow through a

Current is not a point function, but current density is.

finite area (a conducting wire of a finite cross section, for instance); hence it is not a point function. In electromagnetics we define a vector point function ***current density***, **J**, which measures the amount of current flowing through a unit area normal to the direction of current flow. The boldfaced **J** is a vector whose magnitude is the current per unit area (A/m^2) and those direction is the direction of current flow.

Four fundamental electromagnetic field quantities: E, B, D, H

There are four fundamental *vector* field quantities in electromagnetics: ***electric field intensity*** **E**, ***electric flux density*** (or ***electric displacement***) **D**, ***magnetic flux density*** **B**, and ***magnetic field intensity*** **H**. The definition and physical significance of these quantities will be explained fully when they are introduced later in the book. At this time we want only to establish the following. Electric field intensity **E** is the only vector needed in discussing electrostatics (effects of stationary electric charges) in free space; it is defined as the electric force on a unit test charge. Electric displacement vector **D** is useful in the study of electric field in material media, as we shall see in Chapter 3. Similarly, magnetic flux density **B** is the only vector needed in discussing magnetostatics (effects of steady electric currents) in free space and is related to the magnetic force acting on a charge moving with a given velocity. The magnetic field intensity vector **H** is useful in the study of magnetic field in material media. The definition and significance of **B** and **H** will be discussed in Chapter 5.

The four fundamental electromagnetic field quantities, together with their units, are tabulated in Table 1-1. In Table 1-1, V/m is volt per meter, and T stands for tesla or volt-second per square meter. When there is no time variation (as in static, steady, or stationary cases), the electric field quantities **E** and **D** and the magnetic field quantities **B** and **H** form two separate vector pairs. In time-dependent cases, however, electric and magnetic field quantities

TABLE **1-1** FUNDAMENTAL ELECTROMAGNETIC FIELD QUANTITIES

Symbols and Units for Field Quantities	Field Quantity	Symbol	Unit
Electric	Electric field intensity	**E**	V/m
	Electric flux density (Electric displacement)	**D**	C/m^2
Magnetic	Magnetic flux density	**B**	T
	Magnetic field intensity	**H**	A/m

are coupled; that is, time-varying **E** and **D** will give rise to **B** and **H**, and vice versa. All four quantities are point functions. Material (or medium) properties determine the relations between **E** and **D** and between **B** and **H**. These relations are called the ***constitutive relations*** of a medium and will be examined later.

The principal objective of studying electromagnetism is to understand the interaction between charges and currents at a distance based on the electromagnetic model. Fields and waves (time- and space-dependent fields) are basic conceptual quantities of this model. Fundamental postulates, to be enunciated in later chapters, will relate **E**, **D**, **B**, **H** and the source quantities; and derived relations will lead to the explanation and prediction of electromagnetic phenomena.

1-3 SI UNITS AND UNIVERSAL CONSTANTS

The SI, or MKSA, units

A measurement of any physical quantity must be expressed as a number followed by a unit. Thus we may talk about a length of three meters, a mass of two kilograms, and a time period of ten seconds. To be useful, a unit system should be based on some fundamental units of convenient (practical) sizes. In mechanics, all quantities can be expressed in terms of three basic units (for length, mass, and time). In electromagnetics a fourth basic unit (for current) is needed. The ***SI (International System of Units)*** is an ***MKSA system*** built from the four fundamental units listed in Table 1-2. All other units used in electromagnetics, including those appearing in Table 1-1, are derived units expressible in terms of *meters*, *kilograms*, *seconds*, and *amperes*. For example, the unit for charge, coulomb (C), is ampere-second (A · s); the unit for electric field intensity (V/m) is kg · m/A · s^3; and the unit for magnetic flux density, tesla (T), is kg/A · s^2. More complete tables of the units for various quantities are given in Appendix A.

In our electromagnetic model there are three universal constants, in addition to the field quantities listed in Table 1-1. They relate to the

TABLE **1-2** FUNDAMENTAL SI UNITS

Quantity	Unit	Abbreviation
Length	meter	m
Mass	kilogram	kg
Time	second	s
Current	ampere	A

Three universal constants in electromagnetic model

properties of the free space (vacuum). They are as follows: ***velocity of electromagnetic wave*** (including light) in free space, c; ***permittivity*** of free space, ϵ_0; and ***permeability*** of free space, μ_0. Many experiments have been performed for precise measurement of the velocity of light, to many decimal places. For our purpose it is sufficient to remember that

$$\boxed{c \cong 3 \times 10^8 \qquad \text{(m/s).}} \quad \text{(in free space)} \tag{1-6}$$

The other two constants, ϵ_0 and μ_0, pertain to electric and magnetic phenomena, respectively: ϵ_0 is the proportionality constant between the electric flux density **D** and the electric field intensity **E** in free space, such that

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E};} \quad \text{(in free space)} \tag{1-7}$$

μ_0 is the proportionality constant between the magnetic flux density **B** and the magnetic field intensity **H** in free space, such that

$$\boxed{\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}.} \quad \text{(in free space)} \tag{1-8}$$

The values of ϵ_0 and μ_0 are determined by the choice of the unit system, and they are not independent. In the ***SI system***, which is almost universally adopted for electromagnetics work, the permeability of free space is chosen to be

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \qquad \text{(H/m),}} \quad \text{(in free space)} \tag{1-9}$$

where H/m stands for henry per meter. With the values of c and μ_0 fixed in Eqs. (1–6) and (1–9) the value of the permittivity of free space is then derived from the following relationships:

$$\boxed{c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \qquad \text{(m/s),}} \quad \text{(in free space)} \tag{1-10}$$

or

$$\boxed{\begin{aligned} \epsilon_0 &= \frac{1}{c^2 \mu_0} \cong \frac{1}{36\pi} \times 10^{-9} \\ &\cong 8.854 \times 10^{-12} \qquad \text{(F/m),} \end{aligned}} \quad \text{(in free space)} \tag{1-11}$$

TABLE **1-3** UNIVERSAL CONSTANTS IN SI UNITS

Universal Constants	**Symbol**	**Value**	**Unit**
Velocity of light in free space	c	3×10^8	m/s
Permeability of free space	μ_0	$4\pi \times 10^{-7}$	H/m
Permittivity of free space	ϵ_0	$\frac{1}{36\pi} \times 10^{-9}$	F/m

where F/m is the abbreviation for farad per meter. The three universal constants and their values are summarized in Table 1-3.

Now that we have defined the basic quantities and the universal constants of the electromagnetic model, we can develop the various subjects in electromagnetics. But, before we do that, we must be equipped with the appropriate mathematical tools. In the following chapter we discuss the basic rules of operation for vector algebra and vector calculus.

SUMMARY

This chapter laid the foundation for our study of engineering electromagnetism. We adopt a deductive or axiomatic approach and construct an electromagnetic model. Basic source quantities (charge, charge densities, current density) and field quantities (**E**, **D**, **B**, **H**) are defined, the unit system (SI) is specified, and the three universal constants for free space (μ_0, c, ε_0) are given. With this framework we can develop the various topics by introducing the fundamental postulates in subsequent chapters. We shall do this gradually, in steps. But first we need to be familiar with the mathematics that will be used to relate the different quantities. A secure knowledge of vector analysis is essential. Chapter 2 presents the required material on vector algebra and vector calculus.

REVIEW QUESTIONS

Q.1-1 What is electromagnetics?

Q.1-2 Describe two phenomena or situations, other than the mobile telephone depicted in Fig. 1-1, that cannot be adequately explained by circuit theory.

Q.1-3 What are the three essential steps in building an idealized model for the study of a scientific subject?

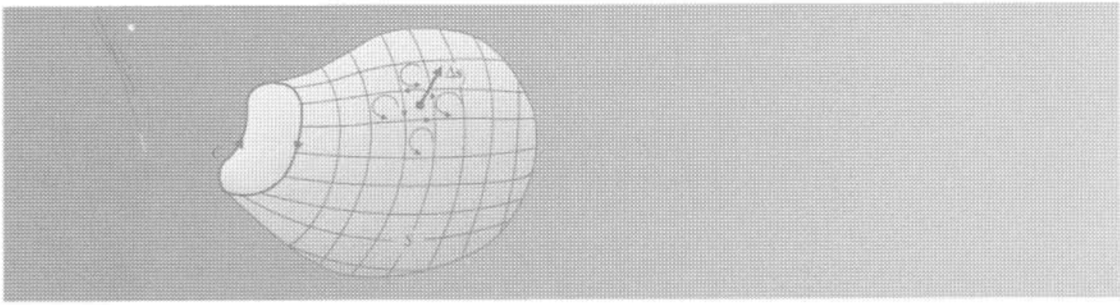
Q.1-4 What are the source quantities in the electromagnetic model?

Q.1-5 What is meant by a *point function*? Is charge density a point function? Is current a point function?

Q.1-6 What are the four fundamental SI units in electromagnetics?

Q.1-7 What are the four fundamental field quantities in the electromagnetic model? What are their units?

Q.1-8 What are the three universal constants in the electromagnetic model, and what are their relations?



CHAPTER 2

2-1 OVERVIEW

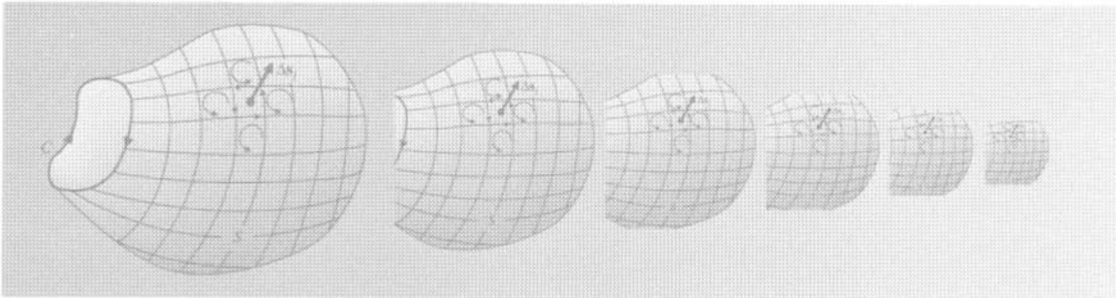
In our electromagnetic model some of the quantities (such as charge, current, and energy) are scalars; and some others (such as electric and magnetic field intensities) are vectors. Both scalars and vectors can be functions of time and position. At a given time and position, a ***scalar*** is completely specified by its magnitude (positive or negative, together with its unit). Thus we can specify, for instance, a charge of $-1(\mu C)$ at a certain location at $t = 0$. The specification of a ***vector*** at a given location and time, on the other hand, requires both a magnitude and a direction. How do we specify the direction of a vector? In a three-dimensional space, three numbers are needed, and these numbers depend on the choice of a coordinate system.

Scalar

Vector

It is important to note that physical laws and theorems relating various scalar and vector quantities must hold irrespective of a coordinate system. *The general expressions of the laws of electromagnetism do not require the specification of a coordinate system.* A particular coordinate system is chosen only when a problem of a given geometry is to be analyzed. For example, if we are to determine the magnetic field at the center of a current-carrying wire loop, it is more convenient to use rectangular coordinates if the loop is rectangular, whereas polar coordinates will be more appropriate if the loop is circular in shape. The basic electromagnetic relation governing the solution of such a problem is the same for both geometries.

Coordinate-system independence



Vector Analysis

Because many electromagnetic quantities are vectors, we must be able to handle (add, subtract, and multiply) them with ease. In order to express specific results in a three-dimensional space, we must choose a suitable coordinate system. In this chapter we will discuss the three most common orthogonal coordinate systems: Cartesian, cylindrical, and spherical coordinates. We will see how to resolve a given vector into components in these coordinates and how to transform from one coordinate system to another.

The use of certain differential operators enables us to express the fundamental postulates and other formulas in electromagnetics in a succinct and general manner. We will discuss the significance of gradient, divergence, and curl operations and prove divergence and Stokes's theorems.

This chapter on vector analysis deals with three main topics:

1. Vector algebra—addition, subtraction, and multiplication of vectors.
2. Orthogonal coordinate systems—Cartesian, cylindrical, and spherical coordinates.
3. Vector calculus—differentiation and integration of vectors; gradient, divergence, and curl operations.

We also prove two important null identities involving repeated applications of differential operators.

2-2 VECTOR ADDITION AND SUBTRACTION

We know that a vector has a magnitude and a direction. A vector $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{a}_A A, \tag{2-1}$$

where A is the magnitude (and has the unit and dimension) of $\mathbf{A}$:

$$A = |\mathbf{A}|, \tag{2-2}$$

which is a scalar. $\mathbf{a}_A$ is a dimensionless unit vector having a unity magnitude; it specifies the direction of $\mathbf{A}$. We can find $\mathbf{a}_A$ from the vector $\mathbf{A}$ by dividing it by its magnitude.

Finding the unit vector from a vector

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}. \tag{2-3}$$

The vector $\mathbf{A}$ can be represented graphically by a directed straight-line segment of a length $|\mathbf{A}| = A$ with its arrowhead pointing in the direction of $\mathbf{a}_A$, as shown in Fig. 2-1.

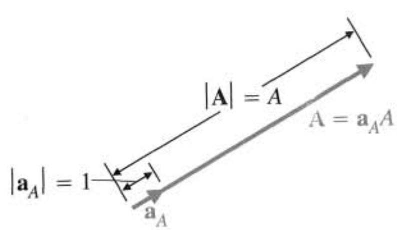
Two vectors are equal if they have the same magnitude and the same direction, even though they may be displaced in space. Since it is difficult to write boldfaced letters by hand, it is a common practice, in writing, to use an arrow or a bar over a letter ($\vec{A}$ or $\bar{A}$) or a wiggly line under a letter ($\underset{\sim}{A}$) to distinguish a vector from a scalar. *This distinguishing mark, once chosen, should never be omitted whenever and wherever vectors are written.*

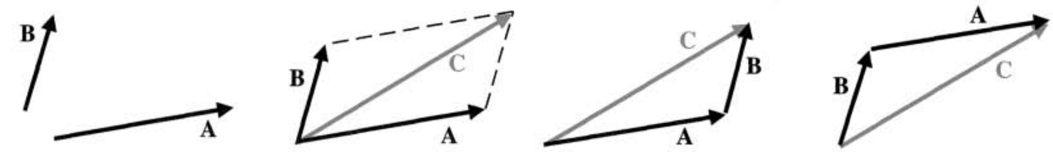
Distinguishing marks for vectors

Two vectors $\mathbf{A}$ and $\mathbf{B}$, which are not in the same direction nor in opposite directions, such as given in Fig. 2-2(a), determine a plane. Their sum is another vector $\mathbf{C}$ in the same plane. $\mathbf{C} = \mathbf{A} + \mathbf{B}$ can be obtained graphically in two ways:

1. By the parallelogram rule: The resultant $\mathbf{C}$ is the diagonal vector of the

FIGURE 2-1 Graphical representation of vector $\mathbf{A}$.





(a) Two vectors, **A** and **B**. (b) Parallelogram rule. (c) Head-to-tail rule, **A** + **B**. (d) Head-to-tail rule, **B** + **A**.

FIGURE 2-2 Vector addition, $\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

parallelogram formed by **A** and **B** drawn from the same point, as shown in Fig. 2-2(b).

2. By the head-to-tail rule: The head of **A** connects to the tail of **B**. Their sum **C** is the vector drawn from the tail of **A** to the head of **B**; and vectors **A**, **B**, and **C** form a triangle, as shown in Fig. 2-2(c). $\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, as illustrated graphically in Fig. 2-2(d).

Vector subtraction can be defined in terms of vector addition in the following way:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}), \tag{2-4}$$

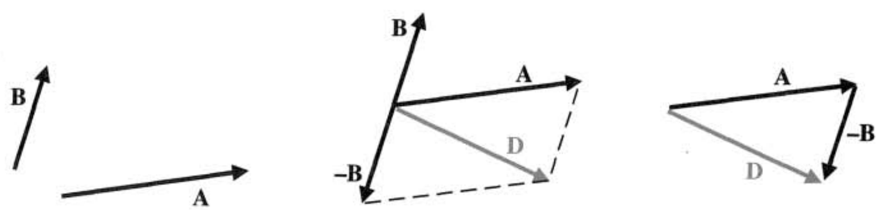
where $-\mathbf{B}$ is the negative of vector **B**. This is illustrated in Fig. 2-3.

NOTE: It is meaningless to add or subtract a scalar from a vector, or to add or subtract a vector from a scalar.

■ **EXERCISE 2.1** Three vectors **A**, **B**, and **C**, drawn in a head-to-tail fashion, form three sides of a triangle. What is $\mathbf{A} + \mathbf{B} + \mathbf{C}$? What is $\mathbf{A} + \mathbf{B} - \mathbf{C}$?

ANS. 0, $-2\mathbf{C}$.

FIGURE 2-3 Vector subtraction, $\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$.



(a) Two vectors, **A** and **B**. (b) Parallelogram rule. (c) Head-to-tail rule.

2-3 VECTOR MULTIPLICATION

Multiplication of a vector **A** by a positive scalar k changes the magnitude of **A** by k times without changing its direction (k can be either greater or less than 1).

$$k\mathbf{A} = \mathbf{a}_A(kA). \tag{2-5}$$

It is not sufficient to say "the multiplication of one vector by another" or "the product of two vectors" because there are two distinct and very different types of products of two vectors. They are (1) the scalar or dot product, and (2) the vector or cross product. These will be defined in the following subsections.

2-3.1 SCALAR OR DOT PRODUCT

The scalar or dot product of two vectors **A** and **B** is denoted by $\mathbf{A}\cdot\mathbf{B}$ ("**A** dot **B**"). The result of the dot product of two vectors is a scalar. It is equal to the product of the magnitudes of **A** and **B** and the cosine of the angle between them. Thus,

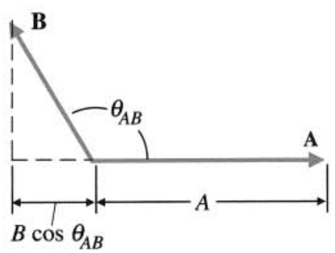
Definition of scalar or dot product of two vectors

$$\boxed{\mathbf{A}\cdot\mathbf{B} \triangleq AB\cos\theta_{AB}.} \tag{2-6}$$

In Eq. (2-6) the symbol $\triangleq$ signifies "equal by definition," and θ_{AB} is the *smaller* angle between **A** and **B** and is less than π radians (180°), as indicated in Fig. 2-4.

From the definition in Eq. (2-6) we see that the dot product of two vectors (1) is less than or equal to the product of their magnitudes; (2) can be either a positive or a negative quantity, depending on whether the angle between them is smaller or larger than $\pi/2$ radians (90°); (3) is equal to the product of the magnitude of one vector and the projection of the other vector

FIGURE 2-4 Illustrating the dot product of **A** and **B**.



upon the first one; and (4) is zero when the vectors are perpendicular to each other.

From Eq. (2-6) we can see that

Dot product is commutative.

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}. \tag{2-7}$$

Thus the order of the vectors in a dot product is not important. (The dot product is commutative.) Also,

$$\mathbf{A}\cdot\mathbf{A} = A^2, \tag{2-8}$$

or

Finding the magnitude of a vector

$$A = |\mathbf{A}| = +\sqrt{\mathbf{A}\cdot\mathbf{A}}. \tag{2-9}$$

Equation (2-9) enables us to find the magnitude of a vector when the expression of the vector is given in *any coordinate system.* We simply form the dot product of the vector with itself, $(\mathbf{A}\cdot\mathbf{A})$, and take the positive square root of the scalar result.

EXAMPLE 2-1

Use vectors to prove the law of cosines for a triangle.

SOLUTION

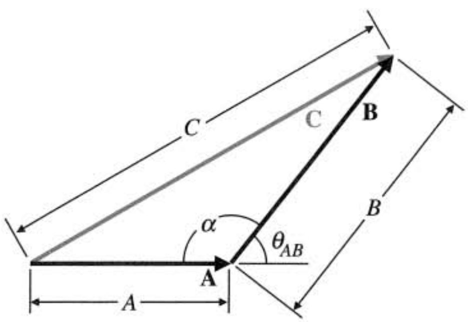
The law of cosines is a scalar relationship that expresses the length of a side of a triangle in terms of the lengths of the two other sides and the angle between them. For Fig. 2-5 the law of cosines states that

$$C = \sqrt{A^2 + B^2 - 2AB\cos\alpha}. \tag{2-10}$$

We prove this by considering the sides as vectors; that is,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}.$$

FIGURE 2-5 Illustrating Example 2-1.



In order to find the magnitude of **C** we take the dot product of **C** with itself, as in Eq. (2-8).

$$\begin{aligned} C^2 = \mathbf{C}\cdot\mathbf{C} &= (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) \\ &= \mathbf{A}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}+2\mathbf{A}\cdot\mathbf{B} \\ &= A^2+B^2+2AB\cos\theta_{AB}. \end{aligned}$$

Since θ_{AB} is, by definition, the *smaller* angle between **A** and **B** and is equal to $(180^\circ - \alpha)$ we know that $\cos\theta_{AB} = \cos(180^\circ - \alpha) = -\cos\alpha$. Therefore,

$$C^2 = A^2 + B^2 - 2AB\cos\alpha. \tag{2-11}$$

The square root of both sides of Eq. (2-11) gives the law of cosines in Eq. (2-10). Note that no coordinate system needs to be specified in this problem.

2-3.2 VECTOR OR CROSS PRODUCT

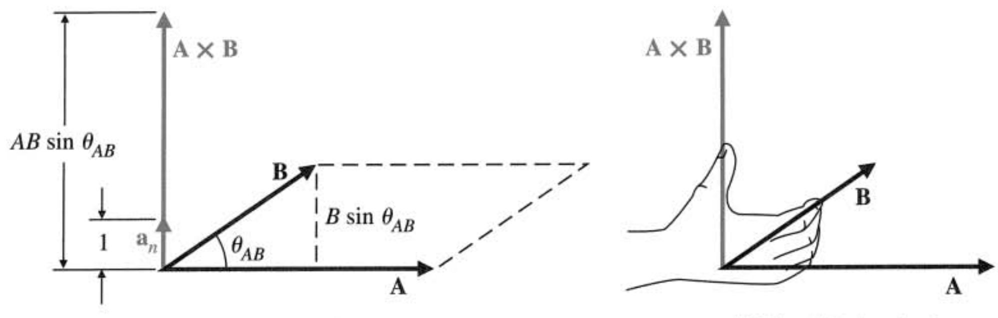
A second kind of vector multiplication is the vector or cross product. Given two vectors **A** and **B**, the cross product, denoted by $\mathbf{A}\times\mathbf{B}$ ("**A** cross **B**"), is another vector defined by

Definition of vector or cross product of two vectors

$$\boxed{\mathbf{A}\times\mathbf{B} \triangleq \mathbf{a}_n AB\sin\theta_{AB},} \tag{2-12}$$

where θ_{AB} is the *smaller* angle between the vectors **A** and **B** ($\leqslant\pi$), and $\mathbf{a}_n$ is a unit vector normal (perpendicular) to the plane containing **A** and **B**. The direction of $\mathbf{a}_n$ follows that of the thumb of a *right hand* when the fingers rotate from **A** to **B** through the angle θ_{AB} (the *right-hand rule*). This is illustrated in Fig. 2-6. From the figure we can see that $B\sin\theta_{AB}$ is the height of the parallelogram formed by the vectors **A** and **B**. We also recognize that the

FIGURE 2-6 Cross product of **A** and **B**, $\mathbf{A}\times\mathbf{B}$.



(a) $\mathbf{A}\times\mathbf{B} = \mathbf{a}_n AB\sin\theta_{AB}$.

(b) The right-hand rule.

quantity $AB \sin \theta_{AB}$, which is nonnegative (positive or 0), is numerically equal to the area of the parallelogram. Thus the cross product $\mathbf{A} \times \mathbf{B}$ results in another vector, whose direction $\mathbf{a}_n$ is obtained by the right-hand rule in rotating from **A** to **B**, and whose magnitude is equal to the area of the parallelogram found by **A** and **B**.

Using the definition in Eq. (2-12) and following the right-hand rule, we find that

Vector product is not commutative.

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}. \tag{2-13}$$

Hence the cross product is *not* commutative; and reversing the order of two vectors in a cross product changes the sign of the product.

2-3.3 PRODUCTS OF THREE VECTORS

There are two kinds of products of three vectors: (1) scalar triple product, and (2) vector triple product.

1. Scalar triple product. This is the dot product of one vector with the result of the cross product of two other vectors. A typical form of this is

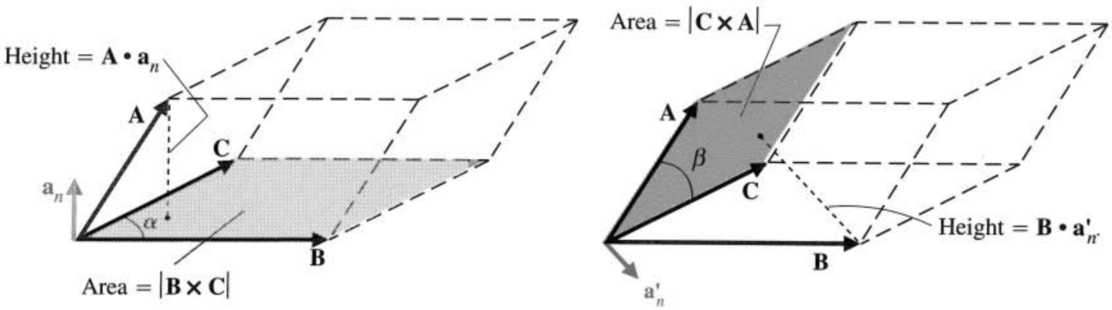
   $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$,

   where **A**, **B**, and **C** are three arbitrary vectors, as illustrated in Fig. 2-7(a).

   According to Eq. (2-12), the cross product $\mathbf{B} \times \mathbf{C}$ has a magnitude $BC \sin \alpha$, which is equal to the area of the shaded parallelogram formed by sides **B** and **C**. The direction of $\mathbf{B} \times \mathbf{C}$ is $\mathbf{a}_n$, a unit normal vector perpendicular to the plane containing **B** and **C**, as shown. The given triple product is then

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{a}_n) BC \sin \alpha. \tag{2-14}$$

FIGURE 2-7 Illustrating scalar triple products.



(a) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. (b) $\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$.

In Eq. (2-14), $(\mathbf{A} \cdot \mathbf{a}_n)$ is a scalar whose magnitude is the projection of **A** in the direction of the unit normal vector $\mathbf{a}_n$. Thus $(\mathbf{A} \cdot \mathbf{a}_n)$ is numerically equal to the height of the parallelepiped formed by the vectors **A**, **B**, and **C**, and the given scalar triple product is equal to the volume of the parallelepiped.

2. Vector triple product. This is the cross product of one vector with the result of the cross product of two other vectors. A typical form of this is

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

This case is more complicated, and we will not attempt a general derivation here. However, it can be expanded quite simply when a coordinate system is given. (See Problem P.2-9.) We will discuss its use when the occasion arises in the future.

EXAMPLE 2-2

Given three vectors **A**, **B**, and **C**, prove the following relation for scalar triple products:

$$\boxed{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).} \tag{2-15}$$

SOLUTION

We have found that the first scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ as expressed in Eq. (2-14) is equal to the volume of the parallelepiped formed by the three vectors **A**, **B**, and **C**. Let us now examine the second scalar triple product $\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$. We have, from Fig. 2-7(b) and Eq. (2-12),

$$\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = (\mathbf{B} \cdot \mathbf{a}'_n) CA \sin \beta, \tag{2-16}$$

where $\mathbf{a}'_n$ and $CA \sin \beta$ represent, respectively, the direction and the magnitude of the cross product $\mathbf{C} \times \mathbf{A}$. Now visualize the parallelepiped formed by the three vectors **A**, **B**, and **C** as standing on the shaded base with an area equal to $|\mathbf{C} \times \mathbf{A}| = CA \sin \beta$. The height of the parallelepiped is $(\mathbf{B} \cdot \mathbf{a}'_n)$. Hence the scalar triple product in Eq. (2-16) has a magnitude equal to the volume of the parallelepiped, which is the same as that given by Eq. (2-14). Thus,

$$\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}). \tag{2-17}$$

Similar arguments apply to the third scalar triple product in Eq. (2-15), $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$, since all three forms yield the volume of the parallelepiped.

CAUTION: The equalities in Eq. (2-15) require that the order of the vectors in the scalar triple product be kept in cyclic permutation. This means that the sequence $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$, $\{\mathbf{B}, \mathbf{C}, \mathbf{A}\}$, or $\{\mathbf{C}, \mathbf{A}, \mathbf{B}\}$ must be maintained when taking

the dot product of the first vector with the result of the cross product of the second and third vectors. $\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C})$, which does not follow the cyclical sequence, is not the same as (but is the negative of) $\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$ in Eq. (2-16).

REVIEW QUESTIONS

Q.2-1 Under what conditions can the dot product of two vectors be negative?

Q.2-2 Write down the results of $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$ if (a) $\mathbf{A}\parallel\mathbf{B}$, and (b) $\mathbf{A}\perp\mathbf{B}$.

Q.2-3 Is $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ equal to $\mathbf{A}(\mathbf{B}\cdot\mathbf{C})$? Explain.

Q.2-4 Given two vectors **A** and **B**, how do you find (a) the component of **A** in the direction of **B**, and (b) the component of **B** in the direction of **A**?

Q.2-5 Does $\mathbf{A}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{C}$ imply $\mathbf{B}=\mathbf{C}$? Explain.

Q.2-6 Does $\mathbf{A}\times\mathbf{B}=\mathbf{A}\times\mathbf{C}$ imply $\mathbf{B}=\mathbf{C}$? Explain.

REMARKS

1. In writing a vector, *never* leave out the mark that distinguishes it from a scalar.
2. Do not add or subtract a vector from a scalar, or vice versa.
3. Division by a vector is not defined. Do not attempt to divide a quantity by a vector.
4. Two vectors are perpendicular to each other if their dot product is zero, and vice versa. ($\theta=\pi/2$, $\cos\theta=0$—Eq. 2-6.)
5. Two vectors are parallel to each other if their cross product is zero, and vice versa. ($\theta=0$, $\sin\theta=0$—Eq. 2-10.)

EXERCISE 2.2 Compare the values of the following scalar triple products of vectors: (a) $(\mathbf{A}\times\mathbf{C})\cdot\mathbf{B}$, (b) $\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B})$, (c) $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$, and (d) $\mathbf{B}\cdot(\mathbf{a}_A\times\mathbf{A})$.

EXERCISE 2.3 Which of the following expressions do not make sense?
(a) $\mathbf{A}\times\mathbf{B}/|\mathbf{B}|$, (b) $\mathbf{C}\cdot\mathbf{D}/(\mathbf{A}\times\mathbf{B})$, (c) $\mathbf{AB}/\mathbf{CD}$, (d) $\mathbf{A}\times\mathbf{B}/(\mathbf{C}\cdot\mathbf{D})$, (e) $\mathbf{ABC}$, (f) $\mathbf{A}\times\mathbf{B}\times\mathbf{C}$.

2-4 ORTHOGONAL COORDINATE SYSTEMS

We have indicated before that although the laws of electromagnetism are invariant with coordinate system, solution of practical problems requires that the relations derived from these laws be expressed in a coordinate system appropriate to the geometry of the given problems. For example, to

determine the electric field at a certain point in space, we at least need to describe the position of the source and the location of this point with respect to a coordinate system. In a three-dimensional space a point can be located as the intersection of three surfaces. Assume that the three families of surfaces are described by u_1 = constant, u_2 = constant, and u_3 = constant, where the u's need not all be lengths, and some may be angles. (In the familiar Cartesian or rectangular coordinate system, u_1, u_2, and u_3 correspond to x, y, and z, respectively.) When these three surfaces are mutually perpendicular to one another, we have an ***orthogonal coordinate system.***

Orthogonal coordinate systems

Many orthogonal coordinate systems exist; but we shall be concerned only with the three that are most common and most useful:

1. Cartesian (or rectangular) coordinates.†
2. Cylindrical coordinates.
3. Spherical coordinates.

These will be discussed separately in the following subsections.

2-4.1 CARTESIAN COORDINATES

A point $P(x_1, y_1, z_1)$ in Cartesian coordinates is the intersection of *three planes* specified by $x = x_1$, $y = y_1$, and $z = z_1$, as shown in Fig. 2-8. We have

$$(u_1, u_2, u_3) = (x, y, z).$$

The three mutually perpendicular unit vectors, $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, in the three coordinate directions are called the ***base vectors.*** For a right-handed system we have the following cyclic properties:

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \tag{2-18a}$$

$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x \tag{2-18b}$$

$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y. \tag{2-18c}$$

The following relations follow directly.

$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_x \cdot \mathbf{a}_z = 0 \tag{2-19}$$

and

$$\mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1. \tag{2-20}$$

The position vector to the point $P(x_1, y_1, z_1)$ is the vector drawn from

†The term "Cartesian coordinates" is preferred because the term "rectangular coordinates" is customarily associated with two-dimensional geometry. The adjective "Cartesian" is used in honor of French philosopher and mathematician Renatus Cartesius (Latinized form for René Descartes 1596–1650), who initiated analytic geometry.

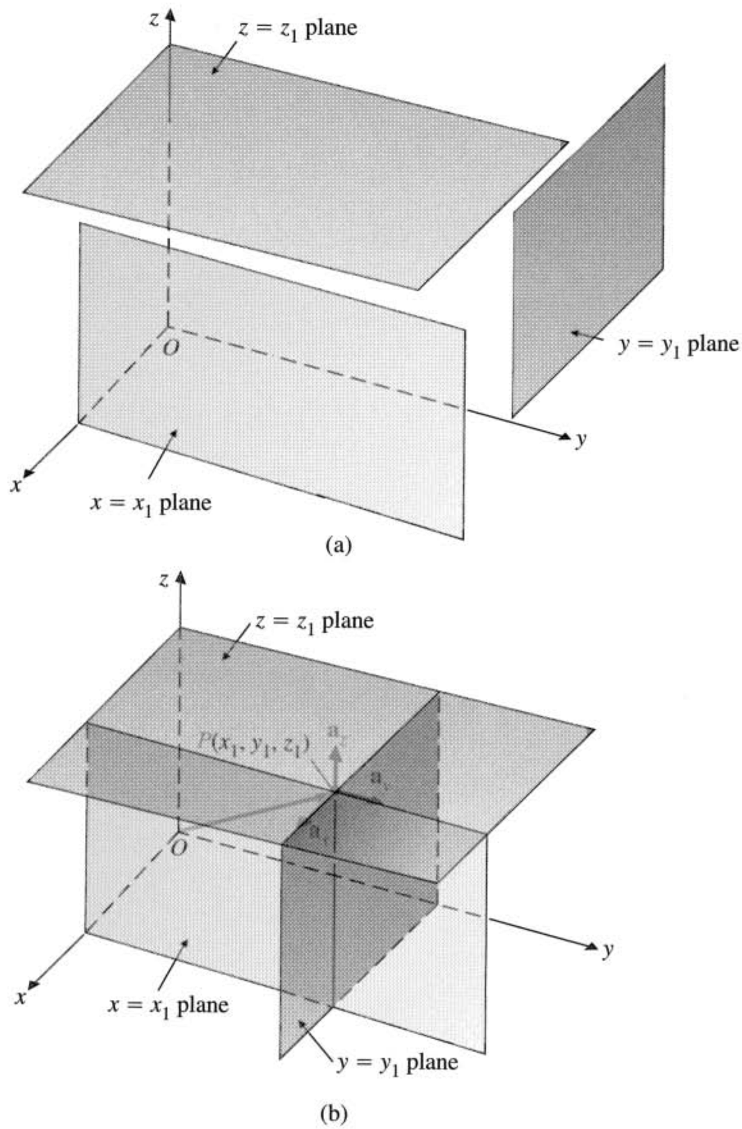


FIGURE 2-8 Cartesian coordinates. (a) Three mutually perpendicular planes. (b) Intersection of the three planes in (a) specifies the location of a point P.

the origin O to P; its components in the $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ directions are, respectively, x_1, y_1, and z_1[†].

$$\overrightarrow{OP} = \mathbf{a}_x x_1 + \mathbf{a}_y y_1 + \mathbf{a}_z z_1. \tag{2-21}$$

[†] In writing vectors in this book we stick to the convention of always writing the direction (of a unit vector) first, followed by the magnitude.

A vector **A** in Cartesian coordinates with components A_x, A_y, and A_z can be written as

Vector A in Cartesian coordinates

$$\boxed{\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z.} \tag{2-22}$$

The expression for a vector differential length is

Vector differential length in Cartesian coordinates

$$\boxed{d\boldsymbol{\ell} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz.} \tag{2-23}$$

A differential volume is the product of the differential length changes in the three coordinate directions:

A differential volume in Cartesian coordinates

$$\boxed{dv = dx\,dy\,dz.} \tag{2-24}$$

The dot product of **A** in Eq. (2-22) and another vector $\mathbf{B} = \mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z$ is

$$\mathbf{A}\cdot\mathbf{B} = (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z)\cdot(\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z),$$

or

Scalar product of A and B in Cartesian coordinates

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z,} \tag{2-25}$$

in view of Eqs. (2-19) and (2-20).

The cross product of **A** and **B** is

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z)\times(\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z)\\ &= \mathbf{a}_x(A_y B_z - A_z B_y) + \mathbf{a}_y(A_z B_x - A_x B_z) + \mathbf{a}_z(A_x B_y - A_y B_x),\end{aligned} \tag{2-26}$$

in view of Eqs. (2-18a, b, and c). Equation (2-26) can be conveniently written in a determinant form for easy memory:

Vector product of A and B in Cartesian coordinates

$$\boxed{\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}.} \tag{2-27}$$

EXAMPLE 2-3

Given a vector $\mathbf{A} = -\mathbf{a}_x + \mathbf{a}_y 2 - \mathbf{a}_z 2$ in Cartesian coordinates, find

a) its magnitude $A = |\mathbf{A}|$,

b) the expression of the unit vector $\mathbf{a}_A$ in the direction of $\mathbf{A}$, and

c) the angle that $\mathbf{A}$ makes with the z-axis.

SOLUTION

a) We find A by using Eqs. (2-8) and (2-9) and noting Eqs. (2-19) and (2-20).

$$\begin{aligned}\mathbf{A}\cdot\mathbf{A} &= (-\mathbf{a}_x + \mathbf{a}_y 2 - \mathbf{a}_z 2)\cdot(-\mathbf{a}_x + \mathbf{a}_y 2 - \mathbf{a}_z 2)\\ &= (-1)(-1) + (2)(2) + (-2)(-2)\\ &= 1 + 4 + 4 = 9.\end{aligned}$$

Thus,

$$A = +\sqrt{\mathbf{A}\cdot\mathbf{A}} = +\sqrt{9} = 3.$$

b) The unit vector $\mathbf{a}_A$ is obtained by using Eq. (2-3). We have

$$\begin{aligned}\mathbf{a}_A = \frac{\mathbf{A}}{A} &= \frac{1}{3}(-\mathbf{a}_x + \mathbf{a}_y 2 - \mathbf{a}_z 2)\\ &= -\mathbf{a}_x\frac{1}{3} + \mathbf{a}_y\frac{2}{3} - \mathbf{a}_z\frac{2}{3}.\end{aligned}$$

c) To find the angle θ_z that $\mathbf{A}$ makes with the $+z$ axis, we take the dot product of $\mathbf{A}$ with the unit vector $\mathbf{a}_z$. From Eq. (2-6) we have

$$\mathbf{A}\cdot\mathbf{a}_z = A\cos\theta_z,$$

$$(-\mathbf{a}_x + \mathbf{a}_y 2 - \mathbf{a}_z 2)\cdot\mathbf{a}_z = -2 = 3\cos\theta_z,$$

from which we obtain

$$\theta_z = \cos^{-1}\left(\frac{-2}{3}\right) = 180^\circ - 48.2^\circ = 131.8^\circ.$$

QUESTION: Why is this answer not -48.2° or 228.2° $(180^\circ + 48.2^\circ)$?

EXAMPLE 2-4

Given $\mathbf{A} = \mathbf{a}_x 5 - \mathbf{a}_y 2 + \mathbf{a}_z$ and $\mathbf{B} = -\mathbf{a}_x 3 + \mathbf{a}_z 4$, find

a) $\mathbf{A}\cdot\mathbf{B}$,

b) $\mathbf{A}\times\mathbf{B}$, and

c) θ_{AB}.

SOLUTION

a) From Eq. (2-25) we find

$$\mathbf{A}\cdot\mathbf{B} = (5)(-3) + (-2)(0) + (1)(4) = -11.$$

b) From Eq. (2-27) we have

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 5 & -2 & 1 \\ -3 & 0 & 4 \end{vmatrix} = -\mathbf{a}_x 8 - \mathbf{a}_y 23 - \mathbf{a}_z 6.$$

c) We can find θ_{AB}, the angle between vectors **A** and **B**, from the definition of $\mathbf{A}\cdot\mathbf{B}$ in Eq. (2-6). The magnitudes A of **A** and B of **B** are:

$$\mathbf{A} = |\mathbf{A}| = +\sqrt{5^2 + (-2)^2 + 1^2} = +\sqrt{30}$$

and

$$\mathbf{B} = |\mathbf{B}| = +\sqrt{(-3)^2 + 4^2} = 5.$$

From Eq. (2-6),

$$\cos\theta_{AB} = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{-11}{5\sqrt{30}} = -0.402.$$

Hence,

$$\theta_{AB} = \cos^{-1}(-0.402) = 180^\circ - 66.3^\circ = 113.7^\circ.$$

EXAMPLE 2-5

a) Write the expression of the vector going from point $P_1(1, 3, 2)$ to point $P_2(3, -2, 4)$ in Cartesian coordinates.

b) Determine the length of the line $\overline{P_1P_2}$.

c) Find the perpendicular distance from the origin to this line.

SOLUTION

a) From Fig. 2-9 we see that

$$\begin{aligned}\overrightarrow{P_1P_2} &= \overrightarrow{OP_2} - \overrightarrow{OP_1} \\ &= (\mathbf{a}_x 3 - \mathbf{a}_y 2 + \mathbf{a}_z 4) - (\mathbf{a}_x + \mathbf{a}_y 3 + \mathbf{a}_z 2) \\ &= \mathbf{a}_x 2 - \mathbf{a}_y 5 + \mathbf{a}_z 2.\end{aligned}$$

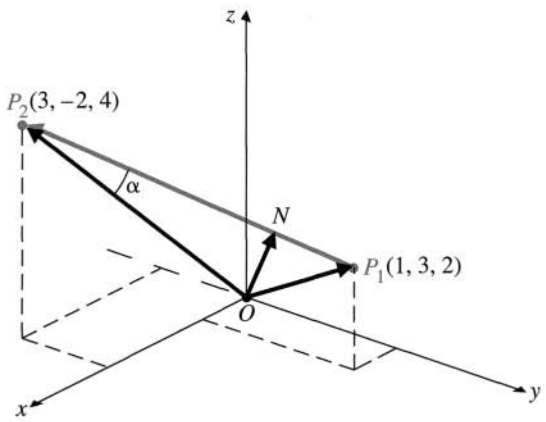


FIGURE 2-9 Illustrating Example 2-5.

b) The length of the line $\overline{P_1P_2}$ is

$$\begin{aligned}\overline{P_1P_2} &= |\overrightarrow{P_1P_2}| \\ &= \sqrt{2^2 + (-5)^2 + 2^2} \\ &= \sqrt{33}.\end{aligned}$$

c) The perpendicular (shortest) distance from the origin O to the line is $|\overrightarrow{ON}|$, which equals $|\overrightarrow{OP_2}|\sin\alpha = |\overrightarrow{OP_2} \times \mathbf{a}_{P_1P_2}|$. Thus,

$$\begin{aligned}|\overrightarrow{ON}| &= \frac{|\overrightarrow{OP_2} \times \overrightarrow{P_1P_2}|}{|\overrightarrow{P_1P_2}|} \\ &= \frac{|(\mathbf{a}_x3 - \mathbf{a}_y2 + \mathbf{a}_z4) \times (\mathbf{a}_x2 - \mathbf{a}_y5 + \mathbf{a}_z2)|}{\sqrt{33}} \\ &= \frac{|\mathbf{a}_x16 + \mathbf{a}_y2 - \mathbf{a}_z11|}{\sqrt{33}} = \frac{\sqrt{381}}{\sqrt{33}} = 3.40.\end{aligned}$$

NOTE: Units have been omitted in this example for simplicity.

EXERCISE 2.4 Given a vector $\mathbf{B} = \mathbf{a}_x2 - \mathbf{a}_y6 + \mathbf{a}_z3$, find

a) the magnitude of $\mathbf{B}$,
b) the expression for $\mathbf{a}_B$,
c) the angles that $\mathbf{B}$ makes with the x, y, and z axes.

ANS. (a) 7, (b) $\mathbf{a}_B = \mathbf{a}_x0.296 - \mathbf{a}_y0.857 + \mathbf{a}_z0.429$, (c) 73.4°, 149.0°, 64.6°.

EXERCISE 2.5 Given two points $P_1(1, 2, 0)$ and $P_2(-3, 4, 0)$ in Cartesian coordinates with origin O, find

a) the length of the projection of $\overrightarrow{OP_2}$ on $\overrightarrow{OP_1}$, and
b) the area of the triangle OP_1P_2.

ANS. (a) 2.236, (b) 5.

2-4.2 CYLINDRICAL COORDINATES

In cylindrical coordinates a point $P(r_1, \phi_1, z_1)$ is the intersection of a circular cylindrical surface $r = r_1$, a half-plane with the z-axis as an edge and making an angle $\phi = \phi_1$ with the xy-plane, and a plane parallel to the xy-plane at $z = z_1$. We have

$$(u_1, u_2, u_3) = (r, \phi, z).$$

As indicated in Fig. 2-10, r is the radial distance measured from the z-axis, and angle ϕ is measured from the positive x-axis. The base vector $\mathbf{a}_\phi$ is tangential to the cylindrical surface. The directions of both $\mathbf{a}_r$ and $\mathbf{a}_\phi$ change with the location of the point P. The following right-hand relations hold for $\mathbf{a}_r$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$:

$$\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z, \tag{2-28a}$$

$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r, \tag{2-28b}$$

$$\mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi. \tag{2-28c}$$

Metric coefficient

Two of the three coordinates, r and z (u_1 and u_3), are themselves lengths. But, $\phi(u_2)$ is an angle, requiring a multiplying coefficient (a *metric coefficient*) r to convert a differential angle change $d\phi$ to a differential length change. This is illustrated in Fig. 2-11.

The metric coefficients for dr and dz are unity. Denoting the three metric coefficients in the three coordinate directions $\mathbf{a}_r$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ by h_1, h_2, and h_3, respectively, we have for cylindrical coordinates $h_1 = 1$, $h_2 = r$, $h_3 = 1$. These are listed in Table 2-1. The metric coefficients in Cartesian coordinates in all three coordinate directions are unity ($h_1 = h_2 = h_3 = 1$), because all three coordinates (x, y, z) are lengths themselves.

The general expression for a vector differential length in cylindrical coordinates is the vector sum of the differential length changes in the three coordinate directions.

Vector differential length in cylindrical coordinates

$$\boxed{d\ell = \mathbf{a}_r dr + \mathbf{a}_\phi r\, d\phi + \mathbf{a}_z dz.} \tag{2-29}$$

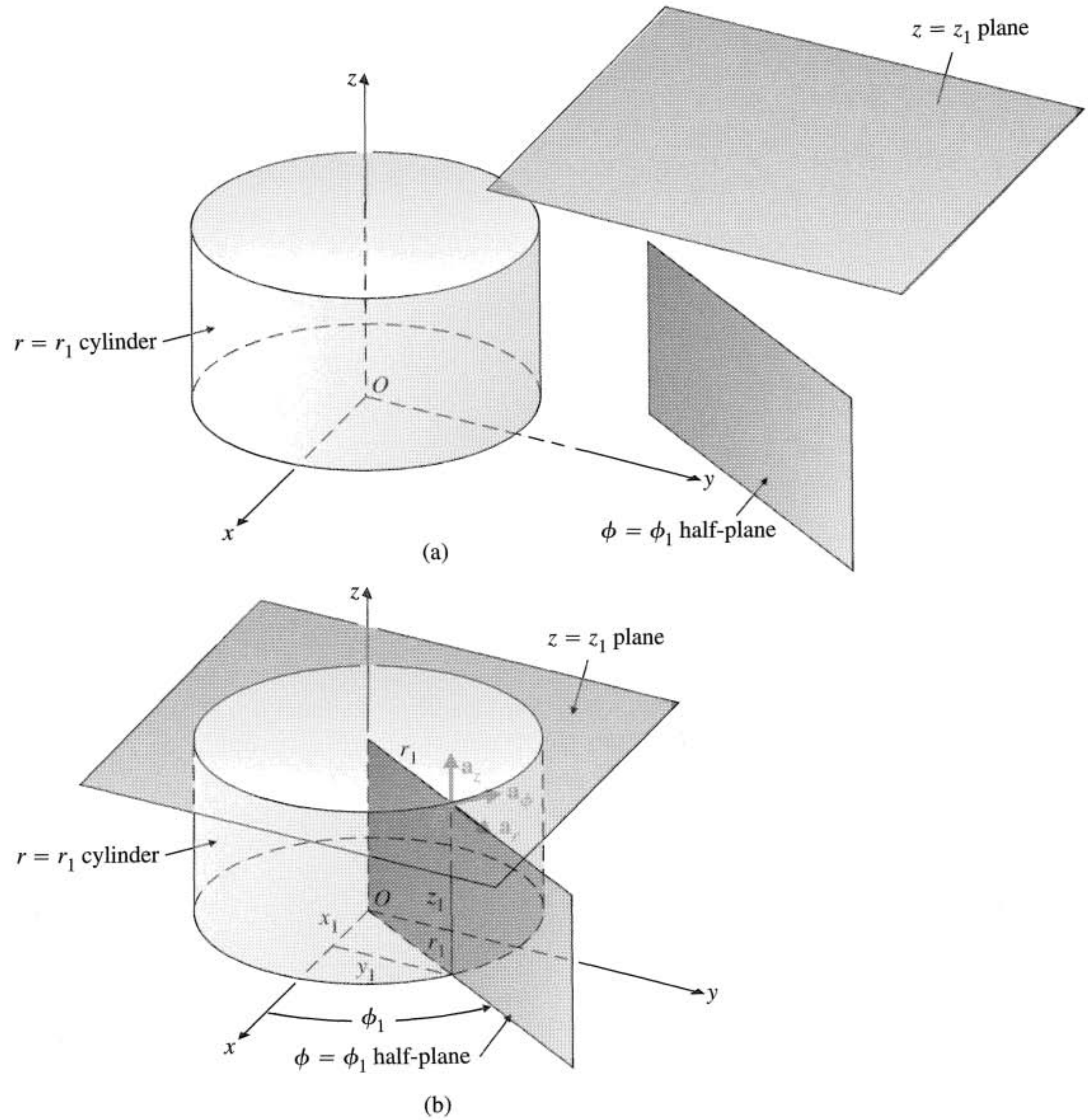


FIGURE 2-10 Cylindrical coördinates. (a) A circular cylindrical surface, a half-plane with the z-axis as an edge, and a plane perpendicular to the z-axis. (b) Intersection of the cylindrical surface and the two planes in (a) specifies the location of a point P.

A differential volume is the product of the differential length changes in the three coordinate directions. In cylindrical coordinates it is

Differential volume in cylindrical coordinates

$$\boxed{dv = r\,dr\,d\phi\,dz.} \tag{2-30}$$

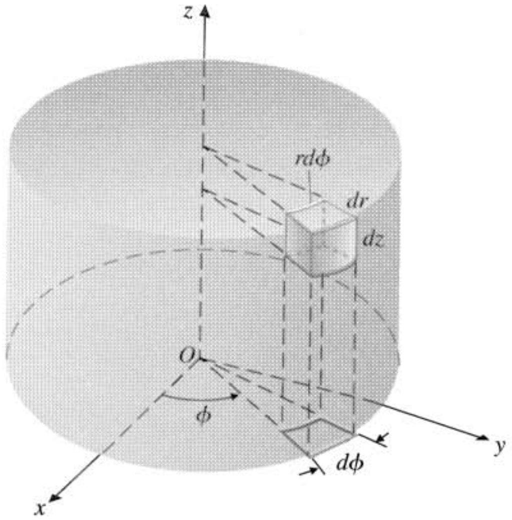


FIGURE 2-11 A differential volume element in cylindrical coordinates.

Cylindrical coordinates are important for problems with long line charges or currents, and in places where cylindrical or circular boundaries exist.

A vector in cylindrical coordinates is written as

Vector A in cylindrical coordinates

$$\boxed{\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z.} \tag{2-31}$$

TABLE **2-1** THREE BASIC ORTHOGONAL COORDINATE SYSTEMS

		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base Vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric Coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R\sin\theta$
Differential Volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Vectors given in cylindrical coordinates can be transformed and expressed in Cartesian coordinates, and vice versa. Suppose we want to express $\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$ in Cartesian coordinates; that is, we want to write $\mathbf{A}$ as $\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$ and determine A_x, A_y, and A_z. First of all, we note that A_z, the z-component of $\mathbf{A}$, is not changed by the transformation from cylindrical to Cartesian coordinates. To find A_x, we equate the dot products of both expressions of $\mathbf{A}$ with $\mathbf{a}_x$. Thus

$$\begin{aligned} A_x &= \mathbf{A} \cdot \mathbf{a}_x \\ &= A_r \mathbf{a}_r \cdot \mathbf{a}_x + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_x. \end{aligned} \tag{2-32}$$

The term containing A_z disappears here because $\mathbf{a}_z \cdot \mathbf{a}_x = 0$. Referring to Fig. 2-12, which shows the relative positions of the base vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_r$, and $\mathbf{a}_\phi$ in the xy-plane, we see that

$$\mathbf{a}_r \cdot \mathbf{a}_x = \cos\phi \tag{2-33}$$

and

$$\mathbf{a}_\phi \cdot \mathbf{a}_x = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi. \tag{2-34}$$

Substituting Eqs. (2-33) and (2-34) into Eq. (2-32), we obtain

$$A_x = A_r \cos\phi - A_\phi \sin\phi. \tag{2-35}$$

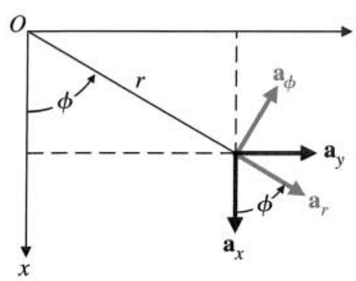
Similarly, to find A_y, we take the dot products of both expressions of $\mathbf{A}$ with $\mathbf{a}_y$:

$$\begin{aligned} A_y &= \mathbf{A} \cdot \mathbf{a}_y \\ &= A_r \mathbf{a}_r \cdot \mathbf{a}_y + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_y. \end{aligned}$$

From Fig. 2-12 we find that

$$\mathbf{a}_r \cdot \mathbf{a}_y = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi \tag{2-36}$$

FIGURE 2-12 Relations among $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_r$, and $\mathbf{a}_\phi$.



and

$$\mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\phi. \tag{2-37}$$

It follows that

$$A_y = A_r \sin\phi + A_\phi \cos\phi. \tag{2-38}$$

It is convenient to write the relations between the components of a vector in Cartesian and cylindrical coordinates in a matrix form:

Transformation of vector components in cylindrical coordinates to Cartesian coordinates

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}.} \tag{2-39}$$

From Fig. 2-12 we see that the coordinates of a point in cylindrical coordinates (r, ϕ, z) can be transformed into those in Cartesian coordinates (x, y, z) as follows:

Transformation of the location of a point in cylindrical coordinates to Cartesian coordinates

$$x = r\cos\phi, \tag{2-40a}$$

$$y = r\sin\phi, \tag{2-40b}$$

$$z = z. \tag{2-40c}$$

EXAMPLE 2-6

Assuming a vector field expressed in cylindrical coordinates to be $\mathbf{A} = \mathbf{a}_r(3\cos\phi) - \mathbf{a}_\phi 2r + \mathbf{a}_z z$,

a) what is the field at the point $P(4, 60^\circ, 5)$?

b) Express the field $\mathbf{A}_P$ at P in Cartesian coordinates.

c) Express the location of the point P in Cartesian coordinates.

SOLUTION

a) At point $P(r = 4, \phi = 60^\circ, z = 5)$ the field is

$$\begin{aligned} \mathbf{A}_P &= \mathbf{a}_r(3\cos 60^\circ) - \mathbf{a}_\phi(2 \times 4) + \mathbf{a}_z 5 \\ &= \mathbf{a}_r(3/2) - \mathbf{a}_\phi 8 + \mathbf{a}_z 5. \end{aligned}$$

b) Using Eq. (2-39), we have

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ -8 \\ 5 \end{bmatrix}$$

$$=\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3/2 \\ -8 \\ 5 \end{bmatrix} = \begin{bmatrix} 7.68 \\ -2.70 \\ 5 \end{bmatrix}.$$

Thus,

$$\mathbf{A}_P = \mathbf{a}_x 7.68 - \mathbf{a}_y 2.70 + \mathbf{a}_z 5.$$

c) Using Eqs. (2-40a, b, and c), we obtain the Cartesian coordinates of the point P as $(4\cos 60°, 4\sin 60°, 5)$, or $(2, 2\sqrt{3}, 5)$.

EXERCISE 2.6 Express the position vector $\overrightarrow{OQ}$ from the origin O to the point $Q(3, 4, 5)$ in cylindrical coordinates.

ANS. $\mathbf{a}_r 5 + \mathbf{a}_z 5$.

EXERCISE 2.7 The cylindrical coordinates of two points P_1 and P_2 are: $P_1(4, 60°, 1)$ and $P_2(3, 180°, -1)$. Determine the distance between these two points.

ANS. $\sqrt{41}$.

2-4.3 SPHERICAL COORDINATES

A point $P(R_1, \theta_1, \phi_1)$ in spherical coordinates is specified as the intersection of the following three surfaces: a spherical surface centered at the origin with a radius $R = R_1$; a right circular cone with its apex at the origin, its axis coinciding with the $+z$-axis and having a half-angle $\theta = \theta_1$; and a half-plane with the z-axis as an edge and making an angle $\phi = \phi_1$ with the xz-plane. We have

$$(u_1, u_2, u_3) = (R, \theta, \phi).$$

$\mathbf{a}_R$ and $\mathbf{a}_r$ are very different.

The three intersecting surfaces are shown in Fig. 2-13. Note that the *base vector* $\mathbf{a}_R$ *at P is radial from the origin and is quite different from* $\mathbf{a}_r$ *in cylindrical coordinates, the latter being perpendicular to the z-axis.* The base vector $\mathbf{a}_\theta$ lies in the $\phi = \phi_1$ plane and is tangential to the spherical surface, whereas the base vector $\mathbf{a}_\phi$ is the same as that in cylindrical coordinates. These are illustrated in Fig. 2-11. For a right-handed system we have

$$\mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi, \tag{2-41a}$$

$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_R, \tag{2-41b}$$

$$\mathbf{a}_\phi \times \mathbf{a}_R = \mathbf{a}_\theta. \tag{2-41c}$$

Spherical coordinates are important for problems involving point sources and regions with spherical boundaries. When an observer is very far from a

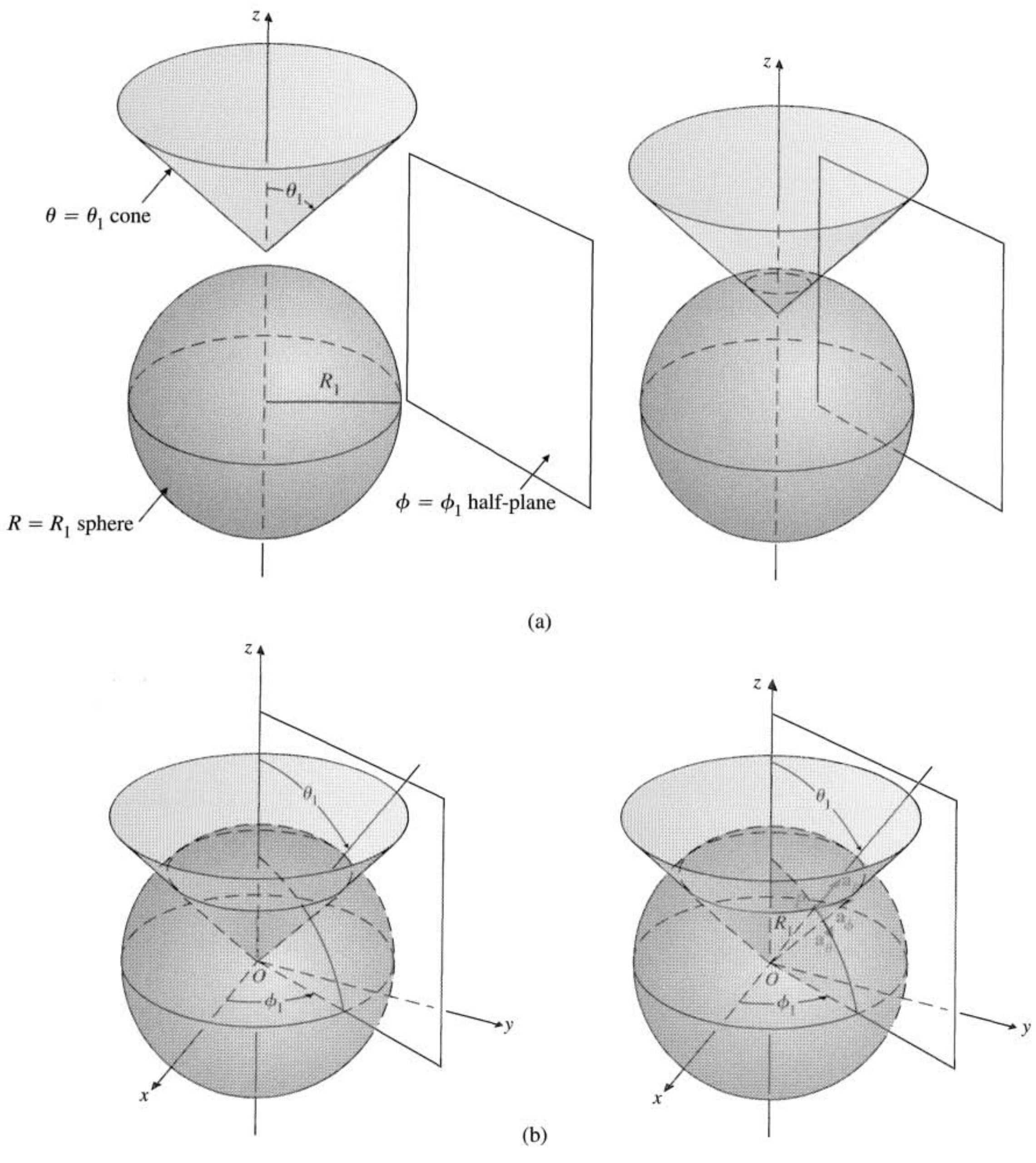


FIGURE 2-13 (a) A spherical surface, a right circular cone, and a half-plane containing the z-axis. (b) Intersection of the sphere, the cone, and the half-plane in (a) specifies the point P.

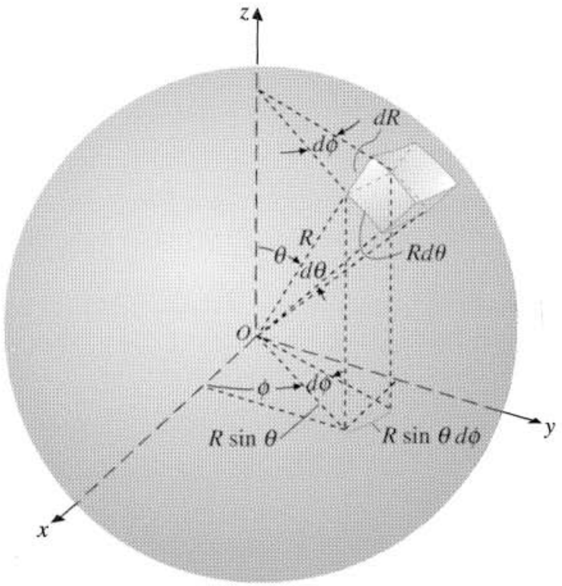


FIGURE 2-14 A differential volume element in spherical coordinates.

source region of a finite extent, the source could be considered approximately as a point. It could be chosen as the origin of a spherical coordinate system so that suitable simplifying approximations could be made. This is the reason that spherical coordinates are used in solving antenna problems in the far field.

A vector in spherical coordinates is written as

Vector A in spherical coordinates

$$\boxed{\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi.} \tag{2-42}$$

In spherical coordinates, only R is a length. The other two coordinates, θ and ϕ are angles. Referring to Fig. 2-14, in which a typical differential volume element is shown, we see that metric coefficients $h_2 = R$ and $h_3 = R \sin\theta$ are required to convert $d\theta$ and $d\phi$, respectively, into differential lengths $(R)d\theta$ and $(R\sin\theta)d\phi$. The general expression for a vector differential length is

Vector differential length in spherical coordinates

$$\boxed{d\ell = \mathbf{a}_R\, dR + \mathbf{a}_\theta R\, d\theta + \mathbf{a}_\phi R \sin\theta\, d\phi.} \tag{2-43}$$

A differential volume is the product of differential length changes in the three coordinate directions:

A differential volume in spherical coordinates

$$\boxed{dv = R^2 \sin\theta \, dR \, d\theta \, d\phi.} \tag{2-44}$$

The base vectors, metric coefficients, and expressions for differential volume for the three basic orthogonal coordinate systems are shown in Table 2-1.

Figure 2-15 shows the interrelationship of the space variables (x, y, z), (r, ϕ, z), and (R, θ, ϕ) that specify the location of a point P. The following equations transform the coordinate variables in spherical coordinates to those in Cartesian coordinates.

Transformation of the location of a point in spherical coordinates to Cartesian coordinates

$$x = R \sin\theta \cos\phi, \tag{2-45a}$$

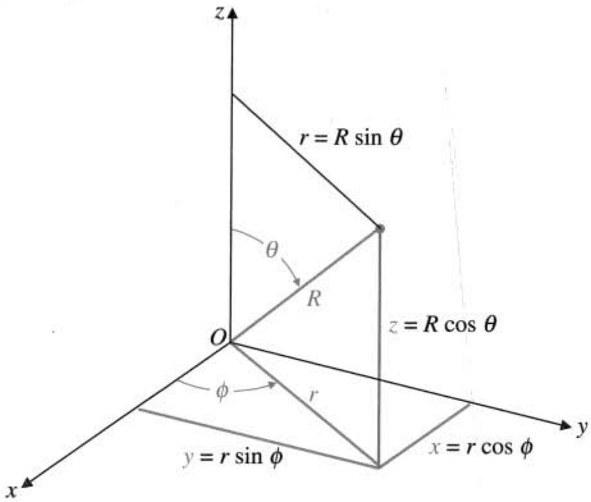
$$y = R \sin\theta \sin\phi, \tag{2-45b}$$

$$z = R \cos\theta. \tag{2-45c}$$

■ **EXERCISE 2.8** Transform Cartesian coordinates $(4, -6, 12)$ into spherical coordinates.

ANS. $(14, 31°, 303.7°)$.

FIGURE 2-15 Showing interrelationship of space variables (x, y, z), (r, ϕ, z), and (R, θ, ϕ).



EXAMPLE 2-7

Express the unit vector $\mathbf{a}_z$ in spherical coordinates.

SOLUTION

First of all, we must not be tempted by Eq. (2-45c) to write $\mathbf{a}_z$ as $\mathbf{a}_R R\cos\theta$ or $\mathbf{a}_R\cos\theta$ because both the direction ($\mathbf{a}_z \neq \mathbf{a}_R$) and the magnitude ($1 \neq R\cos\theta$ or $\cos\theta$ for all θ) would be incorrect. Since the base vectors for spherical coordinates are $\mathbf{a}_R$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$, let us proceed by finding the components of $\mathbf{a}_z$ in these directions. From Figs. 2-13 and 2-14 we have

$$\mathbf{a}_z \cdot \mathbf{a}_R = \cos\theta, \tag{2-46a}$$

$$\mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin\theta, \tag{2-46b}$$

$$\mathbf{a}_z \cdot \mathbf{a}_\phi = 0. \tag{2-46c}$$

Thus,

$$\mathbf{a}_z = \mathbf{a}_R\cos\theta - \mathbf{a}_\theta\sin\theta. \tag{2-47}$$

EXAMPLE 2-8

Assuming that a cloud of electrons confined in a region between two spheres of radii 2 and 5 (cm) has a charge density of

$$\frac{-3\times10^{-8}}{R^4}\cos^2\phi \qquad (\mathrm{C/m^3}),$$

find the total charge contained in the region.

SOLUTION

We have

$$\rho_v = -\frac{3\times10^{-8}}{R^4}\cos^2\phi,$$

$$Q = \int \rho_v\,dv.$$

The given conditions of the problem obviously point to the use of spherical coordinates. Using the expression for dv in Eq. (2-44), we perform a triple integration:

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_{0.02}^{0.05} \rho_v R^2 \sin\theta\, dR\, d\theta\, d\phi.$$